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A difference analogue of the Davey–Stewartson system: discrete Gram-type determinant solution and Lax pair

Gegenhasi[1,2,3], Xing-Biao Hu[2], Decio Levi[4] and Satoshi Tsujimoto[5]

[1] Department of Mathematics, China University of Mining and Technology, Beijing 100083, People's Republic of China
[2] Institute of Computational Mathematics and Scientific Engineering Computing, AMSS, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China
[3] Graduate School of the Chinese Academy of Sciences, Beijing, People's Republic of China
[4] Dipartimento di Ingegneria Elettronica, Universitá degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale, 84 00146 Roma, Italy
[5] Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

E-mail: gegen@amss.ac.cn, hxb@lsec.cc.ac.cn, levi@roma3.infn.it and tujimoto@i.kyoto-u.ac.jp

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Abstract
We consider a difference–difference Davey–Stewartson system together with its bilinear structure. We write some new Gram-type determinantal solutions taking into account a set of Jacobi identities for determinants. A bilinear Bäcklund transformation is constructed and consequently a Lax pair for the discrete system is derived.

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1. Introduction

Integrable systems with both spatial and temporal discretizations play an increasingly important role in the recent development of integrable system theory. Recent research on integrable discrete systems has uncovered various aspects of their properties, such as the relation with numerical algorithms or with Painlevé equations [1–14].

Integrable difference analogues of the differential soliton equations have been introduced using, among others, inverse scattering methods [1], bilinear methods [15–19], transformation group theoretic methods [20].

In [18], the authors have proposed an integrable differential-difference Davey–Stewartson (DS) system:

$$\begin{aligned}
&\mathrm{i}v_t + \alpha_1\,\mathrm{e}^{u_{n-1}+u_{n+1}-2u}v_{n-1} + \alpha_2\,\mathrm{e}^{u_{k-1}+u_{k+1}-2u}v_{k+1} - (\alpha_1+\alpha_2)v = 0,\\
&-\mathrm{i}w_t + \alpha_1\,\mathrm{e}^{u_{n-1}+u_{n+1}-2u}w_{n+1} + \alpha_2\,\mathrm{e}^{u_{k-1}+u_{k+1}-2u}w_{k-1} - (\alpha_1+\alpha_2)w = 0,\\
&z_1 - z_1\,\mathrm{e}^{u_{n+1,k+1}+u-u_{k+1}-u_{n+1}} + z_2 v_{k+1}w_{n+1} = 0,
\end{aligned}\tag{1.1}$$

where α_1, α_2, z_1 and z_2 are the constants. System (1.1) is integrable as it possesses soliton solutions expressed in terms of determinants, a Bäcklund transformation and a Lax pair. By the dependent variable transformation

$$u = \ln F, \qquad v = \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)t}G/F, \qquad w = \mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)t}H/F, \tag{1.2}$$

equations (1.1) are transformed into the bilinear form

$$\begin{aligned}
&[\mathrm{i}D_t + \alpha_1\,\mathrm{e}^{-D_n} + \alpha_2\,\mathrm{e}^{D_k}]G\cdot F = 0,\\
&[\mathrm{i}D_t + \alpha_1\,\mathrm{e}^{-D_n} + \alpha_2\,\mathrm{e}^{D_k}]F\cdot H = 0,\\
&z_1[\mathrm{e}^{1/2(D_n-D_k)} - \mathrm{e}^{1/2(D_n+D_k)}]F\cdot F + z_2\,\mathrm{e}^{1/2(D_k-D_n)}G\cdot H = 0,
\end{aligned}\tag{1.3}$$

where, as usual, the bilinear operators D_t and $\exp(\delta D_n)$ [19] are defined as

$$\begin{aligned}
&D_t^m a\cdot b \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(t)b(t')\bigg|_{t'=t},\\
&\exp(\delta D_n)a\cdot b \equiv a(n+\delta)b(n-\delta).
\end{aligned}$$

In [18], one has derived the bilinear Bäcklund transformation for the bilinear system (1.3), whose space part is given by

$$\begin{aligned}
&[\beta_1\,\mathrm{e}^{1/2D_n} - \mathrm{e}^{-1/2D_n} - \mu_1\,\mathrm{e}^{D_m-1/2D_n}]f\cdot g = 0,\\
&[\beta_2\,\mathrm{e}^{1/2(D_m+D_k)} - \mathrm{e}^{-1/2(D_m+D_k)} - \mu_2\,\mathrm{e}^{1/2(D_m-D_k)}]f\cdot g = 0,
\end{aligned}\tag{1.4}$$

with $\beta_1, \beta_2, \mu_1, \mu_2$ being arbitrary constants and where the function $f(m,n,k;t)$ is obtained from F, G, H by introducing an additional discrete variable m,

$$\begin{aligned}
F(n,k;t) = f(m,n,k;t), \qquad G(n,k;t) = f(m+1,n,k;t),\\
H(n,k;t) = f(m-1,n,k;t),
\end{aligned}$$

and similarly for $g(m,n,k;t)$.

Through the dependent variable transformation,

$$u = \ln f, \qquad v = \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)t}f_{m+1}/f, \qquad w = \mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)t}f_{m-1}/f, \tag{1.5}$$

$$u' = \ln g, \qquad v' = \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)t}g_{m+1}/g, \qquad w' = \mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)t}g_{m-1}/g, \tag{1.6}$$

equations (1.4) are transformed into the nonlinear difference system

$$\frac{v'_{k+1}+\mu_2 v}{v_{k+1}} = \frac{w+\mu_2 w'_{k+1}}{w'} \tag{1.7}$$

$$\frac{v'_{k+1}+\mu_2 v}{v_{k+1}}(1+\mu_1 v_{k+1}w'_{n+1,k+1}) = \frac{v'_{k+1,n+1}+\mu_2 v_{n+1}}{v_{k+1,n+1}}(1+\mu_1 v w'_{n+1}), \tag{1.8}$$

$$\beta\,\mathrm{e}^{u_{n+1}-u'_{n+1}+u'-u} = 1+\mu_1 v w'_{n+1}. \tag{1.9}$$

In equations (1.5)–(1.9) and in the following we will always use a simplified notation for the functions $f(s,\tilde{s})$. We explicitly write a discrete independent variable only when it is shifted from its position. For example,

$$f \equiv f(s,\tilde{s}), \qquad f_{\tilde{s}+1} \equiv f(s,\tilde{s}+1), \qquad f_{s+1,\tilde{s}+1} \equiv f(s+1,\tilde{s}+1).$$

Equations (1.4) can be thought of as equations defining a function $f(m,n,k;t)$ where the t dependence is given by requiring that $g(m,n,k;t) = f(m,n,k;t-\delta)$. In this way, the Bäcklund transformation for a differential-difference equation will provide us with a discretization of the differential-difference equation [21]. Applying the same definition to the functions u, v, w, equations (1.7)–(1.9) become the difference–difference system

$$\frac{v_{k+1}^{t-\delta}+\mu_2 v}{v_{k+1}} = \frac{w+\mu_2 w_{k+1}^{t-\delta}}{w^{t-\delta}} \tag{1.10}$$

$$\frac{v_{k+1}^{t-\delta}+\mu_2 v}{v_{k+1}}\left(1+\mu_1 v_{k+1} w_{n+1,k+1}^{t-\delta}\right) = \frac{v_{k+1,n+1}^{t-\delta}+\mu_2 v_{n+1}}{v_{k+1,n+1}}\left(1+\mu_1 v w_{n+1}^{t-\delta}\right), \tag{1.11}$$

$$\beta\,\mathrm{e}^{u_{n+1}-u_{n+1}^{t-\delta}+u^{t-\delta}-u} = 1+\mu_1 v w_{n+1}^{t-\delta}, \tag{1.12}$$

which has the same bilinear system as the modified discrete KP equation

$$[\beta_1\,\mathrm{e}^{1/2(D_{k_3}-D_{k_2})} - \mathrm{e}^{-1/2(D_{k_3}-D_{k_2})} - \mu_1\,\mathrm{e}^{1/2(D_{k_2}+D_{k_3})}]f\cdot g = 0, \tag{1.13}$$

$$[\beta_2\,\mathrm{e}^{1/2(D_{k_3}-D_{k_1})} - \mathrm{e}^{-1/2(D_{k_3}-D_{k_1})} - \mu_2\,\mathrm{e}^{1/2(D_{k_1}+D_{k_3})}]f\cdot g = 0, \tag{1.14}$$

under the dependent variable transformations $k_1 = -k$, $k_2 = m-n$ and $k_3 = m$.

In this paper, we investigate a different integrable time discretization of the bilinear system (1.3) given by

$$\begin{aligned}
&\left[\mathrm{i}\frac{1}{\delta_1}\sinh(\delta_1 D_t) + \alpha_1\,\mathrm{e}^{-D_n-\delta_1 D_t} + \alpha_2\,\mathrm{e}^{D_k+\delta_1 D_t}\right] G\cdot F = 0,\\
&\left[\mathrm{i}\frac{1}{\delta_1}\sinh(\delta_1 D_t) + \alpha_1\,\mathrm{e}^{-D_n-\delta_1 D_t} + \alpha_2\,\mathrm{e}^{D_k+\delta_1 D_t}\right] F\cdot H = 0,\\
&z_1[\mathrm{e}^{1/2(D_n-D_k)} - \mathrm{e}^{1/2(D_n+D_k)}]F\cdot F + z_2\,\mathrm{e}^{1/2(D_k-D_n)} G\cdot H = 0.
\end{aligned} \tag{1.15}$$

By setting $n=\tilde{s}, k=s, z_1=z_2=1, \delta=\frac{\delta_1}{2}$ and defining the new constants $b_1 = -2\mathrm{i}\alpha_1\delta_1$, $a_1 = 2\mathrm{i}\alpha_2\delta_1$, equations (1.15) can be written as

$$\begin{aligned}
&F_{s+1,\tilde{s}+1}F - F_{s+1}F_{\tilde{s}+1} - G_{s+1}H_{\tilde{s}+1} = 0,\\
&FG^{t+\delta} - F^{t+\delta}G - a_1 F_{s-1}G_{s+1}^{t+\delta} + b_1 F_{\tilde{s}+1}^{t+\delta}G_{\tilde{s}-1} = 0,\\
&HF^{t+\delta} - H^{t+\delta}F - a_1 H_{s-1}F_{s+1}^{t+\delta} + b_1 H_{\tilde{s}+1}^{t+\delta}F_{\tilde{s}-1} = 0.
\end{aligned} \tag{1.16}$$

In [16], the system of equations (1.16) is derived as a specialization of the discrete two-component KP hierarchy. It was also shown there that a continuum limit of the difference equations (1.16) gives the bilinear form of the DS equation. Moreover, by the dependent variable transformation

$$u = \frac{G}{F}, \qquad \tilde{u} = \frac{H}{F}, \qquad v = \frac{F_{s+1}^{t+\delta}F_{s-1}}{F^{t+\delta}F}, \qquad \tilde{v} = \frac{F_{\tilde{s}+1}^{t+\delta}F_{\tilde{s}-1}}{F^{t+\delta}F}, \tag{1.17}$$

equations (1.16) are transformed into the nonlinear difference system

$$\begin{aligned}
&u^{t+\delta} - u - a_1 u_{s+1}^{t+\delta} v + b_1 u_{\tilde{s}-1}\tilde{v} = 0,\\
&\tilde{u}^{t+\delta} - \tilde{u} - b_1 \tilde{u}_{\tilde{s}+1}^{t+\delta}\tilde{v} + a_1\tilde{u}_{s-1} v = 0,\\
&\frac{v_{\tilde{s}+1}}{v} = \frac{1+u_{s+1}^{t+\delta}\tilde{u}_{\tilde{s}+1}^{t+\delta}}{1+u\tilde{u}_{s-1,\tilde{s}+1}},\\
&\frac{\tilde{v}_{s+1}}{\tilde{v}} = \frac{1+u_{s+1}^{t+\delta}\tilde{u}_{\tilde{s}+1}^{t+\delta}}{1+u_{s+1,\tilde{s}-1}\tilde{u}}.
\end{aligned} \tag{1.18}$$

The bilinear differential-difference DS system (1.3) has solutions represented by the Casorati- and Gram-type determinants [18]. In [16] one can find the discrete bilinear structure of difference analogue of the DS system (1.18) and their solutions expressed in terms of double

Casorati determinants. In this paper, we present solutions for the difference equations (1.18) written in terms of discrete Gram-type determinants.

This paper is organized as follows. In section 2, we first review the double-Casorati determinant solutions to the difference system (1.16) obtained in [16], and then in section 3 we will present its discrete Gram-type determinant solutions. A bilinear Bäcklund transformation for the discrete system (1.16) is then constructed in section 4 and consequently a discrete Lax pair is derived. Section 5 is devoted to the conclusions and a discussion of the results obtained.

2. Casorati determinant solutions to the difference analogue of the DS system

It is known that there are two types of determinant representations for the solutions of the discrete bilinear equations. One is the Casorati determinant solution and the other is the discrete Gram-type determinant solution [15]. In [16], one can find the following double-Casorati determinant solutions for the difference DS system (1.16) while the double-Wronski determinant solutions and double-Casorati determinant solutions for the DS system and differential-difference DS system were given in [18, 22]. Defining

$$F = \tau_m^t(s,\tilde{s}), \qquad G = \tau_{m+1}^t(s-1,\tilde{s}+1), \qquad H = \tau_{m-1}^t(s+1,\tilde{s}-1), \tag{2.1}$$

the solutions to equations (1.16) are given by

$$\begin{aligned}\tau_m^t(s,\tilde{s}) &\equiv \tau_m^{k,l}(s,\tilde{s})\\ &= \begin{vmatrix} \phi_1^k(s) & \cdots & \phi_1^k(s+m-1); & \psi_1^l(\tilde{s}) & \cdots & \psi_1^l(\tilde{s}+2N-m-1)\\ \phi_2^k(s) & \cdots & \phi_2^k(s+m-1); & \psi_2^l(\tilde{s}) & \cdots & \psi_2^l(\tilde{s}+2N-m-1)\\ \vdots & & \vdots & \vdots & & \vdots\\ \phi_{2N}^k(s) & \cdots & \phi_{2N}^k(s+m-1); & \psi_{2N}^l(\tilde{s}) & \cdots & \psi_{2N}^l(\tilde{s}+2N-m-1)\end{vmatrix}\\ &= |0,1,\ldots,m-1;0',1',\ldots,(M-m-1)'|,\end{aligned} \tag{2.2}$$

where

$$\tau_m^{t+\delta}(s,\tilde{s}) = \tau_m^{k+a_1,l+b_1}(s,\tilde{s}), \tag{2.3}$$

$\phi_r^k(s)$ and $\psi_r^l(\tilde{s})(r = 1, 2, \ldots, 2N)$ satisfy the linear difference equations:

$$\phi_r - \phi_r^{k-a_1} = a_1\phi_r(s+1), \tag{2.4}$$

$$\psi_r - \psi_r^{l-b_1} = b_1\psi_r(\tilde{s}+1). \tag{2.5}$$

The proof is obtained by substituting equation (2.1) with equations (2.2)–(2.5) into equations (1.16), and taking into account that the following determinant identities are satisfied:

$$\begin{aligned}&|0,1,\ldots,m-1;0',1',\ldots,(M-m-1)'||1,2,\ldots,m;1',2',\ldots,(M-m)'|\\ &-|1,2,\ldots,m;0',1',\ldots,(M-m-1)'||0,1,\ldots,m-1;1',2',\ldots,(M-m)'|\\ &-|1,2,\ldots,m-1;0',1',\ldots,(M-m)'||0,1,\ldots,m;1',\ldots,(M-m-1)'|\\ &=0,\end{aligned}$$

$$\begin{aligned}&|0,1,\ldots,m-1;0',1',\ldots,(M-m-1)'|\\ &|-1,0,\ldots,m-2,(m-1)|_{k+a_1};1',2',\ldots,(M-m-2)',(M-m-1)'|_{l+b_1}|\\ &-|0,1,\ldots,m-2,(m-1)|_{k+a_1};0',1',\ldots,(M-m-2)',(M-m-1)'|_{l+b_1}|\\ &|-1,0,\ldots,m-1;1',2',\ldots,(M-m-1)'|\\ &-|0,1,\ldots,m-1,(m-1)|_{k+a_1};0',1',\ldots,(M-m-2)',(M-m-1)'|_{l+b_1}|\\ &|-1,0,\ldots,m-2;1',\ldots,(M-m-1)'|\end{aligned}$$

$$+|0, 1, \ldots, m-2, (m-1)|_{k+a_1}; 1', 2', \ldots, (M-m-1)', (M-m-1)'|_{l+b_1}|$$
$$|-1, 0, \ldots, m-1; 0', 1', \ldots, (M-m-2)'|$$
$$= 0,$$

$$|1, 2, \ldots, m-1; (-1)', 0', \ldots, (M-m-1)'|$$
$$|0, 1, \ldots, m-2, (m-1)|_{k+a_1}; 0', 1', \ldots, (M-m-2)', (M-m-1)'|_{l+b_1}|$$
$$-|1, 2, \ldots, m-2, (m-1)|_{k+a_1}; (-1)', 0', \ldots, (M-m-2)', (M-m-1)'|_{l+b_1}|$$
$$|0, 1, \ldots, m-1; 0', 1', \ldots, (M-m-1)'|$$
$$-|1, 2, \ldots, m-1, (m-1)|_{k+a_1}; 0', 1', \ldots, (M-m-2)', (M-m-1)'|_{l+b_1}|$$
$$|0, 1, \ldots, m-2; (-1)', 0', \ldots, (M-m-1)'|$$
$$+|1, 2, \ldots, m-2, (m-1)|_{k+a_1}; 0', 1', \ldots, (M-m-1)', (M-m-1)'|_{l+b_1}|$$
$$|0, 1, \ldots, m-1; (-1)', 0', \ldots, (M-m-2)'|$$
$$= 0,$$

where $(m-1)|_{k+a_1}$ and $(M-m-1)'|_{l+b_1}$ denote the column vectors $\big(\phi_1^{k+a_1}(s+m-1), \phi_2^{k+a_1}(s+m-1), \ldots, \phi_{2N}^{k+a_1}(s+m-1)\big)^T$ and $\big(\phi_1^{l+b_1}(s+M-m-1), \phi_2^{l+b_1}(s+M-m-1), \ldots, \phi_{2N}^{l+b_1}(s+M-m-1)\big)^T$, respectively.

3. Discrete Gram-type determinant solutions to the difference analogue of the DS system

In [18, 23], the Gram determinant solution and discrete Gram-type determinant solution for the DS system and differential-difference DS system were obtained. Here, the discrete Gram-type determinant solution to the difference system (1.16) is given by the following proposition:

Proposition 1. *The functions*

$$F = |C + \Omega| = |\mathbf{F}|, \tag{3.1}$$

$$G = -\begin{vmatrix} \mathbf{F} & \Phi(s-1) \\ \Psi'(-\tilde{s}-1)^T & 0 \end{vmatrix}, \qquad H = \begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) \\ \Phi'(-s-1)^T & 0 \end{vmatrix}, \tag{3.2}$$

where $\mathbf{F}$ *is an* $(M+N)\times(M+N)$ *matrix,* $C = (c_{\mu\nu})$ *is an* $(M+N)\times(M+N)$ *constant matrix,* Ω *is an* $(M+N)\times(M+N)$ *block diagonal matrix*

$$\Omega = \begin{pmatrix} m_{ij} & \\ & n_{i'j'} \end{pmatrix},$$

and Φ, Φ', Ψ, Ψ' *are* $M+N$ *column vectors*

$$\Phi = (\varphi_1(s), \ldots, \varphi_M(s); 0, \ldots, 0)^T,$$
$$\Phi' = (\varphi_1'(-s), \ldots, \varphi_M'(-s); 0, \ldots, 0)^T,$$
$$\Psi = (0, \ldots, 0; \psi_1(\tilde{s}), \ldots, \psi_N(\tilde{s}))^T,$$
$$\Psi' = (0, \ldots, 0; \psi_1'(-\tilde{s}), \ldots, \psi_N'(-\tilde{s}))^T,$$

with $m_{ij}, n_{i'j'}, \varphi_i(s,k), \varphi_i'(-s,k), \psi_{i'}(\tilde{s},l), \psi_{i'}'(-\tilde{s},l), i,j \in \{1,\ldots,M\}, i',j' \in \{1,\ldots,N\}$*, satisfying the following difference equations:*

$$m_{ij}^{s+1} = m_{ij} - \phi_i\phi_j'(-s-1), \qquad m_{ij}^{k+a_1} = m_{ij} - a_1\phi_i(k+a_1)\phi_j', \tag{3.3}$$

$$n_{ij}^{\tilde{s}+1} = n_{ij} - \psi_i\psi_j'(-\tilde{s}-1), \qquad n_{ij}^{l+b_1} = n_{ij} - b_1\psi_i(l+b_1)\psi_j', \tag{3.4}$$

$$\phi_i^{k-a_1} = \phi_i - a_1\phi_i(s+1), \qquad \phi_j'^{k+a_1} = \phi_j' - a_1\phi_j'(-s+1), \tag{3.5}$$

$$\psi_j^{l-b_1} = \psi_j - b_1\psi_j(\tilde{s}+1), \qquad \psi_j'^{l+b_1} = \psi_j - b_1\psi_j'(-\tilde{s}+1), \tag{3.6}$$

solve equations (1.16). Here a_1 and b_1 are the arbitrary constants.

Proof. Using equations (3.3)–(3.6), we are able to express, after some calculations, the functions appearing in equations (1.16) in terms of the discrete Gram-type determinants:

$$F_{s+1} = \begin{vmatrix} \mathbf{F} & \Phi \\ \Phi'(-s-1)^T & 1 \end{vmatrix}, \qquad F_{\tilde{s}+1} = \begin{vmatrix} \mathbf{F} & \Psi \\ \Psi'(-\tilde{s}-1)^T & 1 \end{vmatrix},$$

$$G_{s+1} = \begin{vmatrix} \mathbf{F} & \Phi \\ \Psi'(-\tilde{s}-1)^T & 0 \end{vmatrix}, \qquad H_{\tilde{s}+1} = \begin{vmatrix} \mathbf{F} & \Psi \\ \Phi'(-s-1)^T & 0 \end{vmatrix},$$

$$F_{s+1,\tilde{s}+1} = \begin{vmatrix} \mathbf{F} & \Phi & \Psi \\ \Phi'(-s-1)^T & 1 & 0 \\ \Psi'(-\tilde{s}-1)^T & 0 & 1 \end{vmatrix}, \qquad F_{\tilde{s}+1}^{t+\delta} = \begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & \Psi^{l+b_1} \\ \Phi'^T & 1 & 0 \\ \Psi'(-\tilde{s}-1)^T & 0 & 1 \end{vmatrix}$$

$$F^{t+\delta} = \begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'^T & 1 & 0 \\ \Psi'^T & 0 & 1 \end{vmatrix}, \qquad G_{s+1}^{t+\delta} = \begin{vmatrix} \mathbf{F} & \Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Psi'(-\tilde{s}-1)^T & 0 & b_1 \\ \Psi'^T & 0 & 1 \end{vmatrix}$$

$$F_{s-1} = \begin{vmatrix} \mathbf{F} & -\Phi(s-1) \\ \Phi'^T & 1 \end{vmatrix}, \qquad G_{\tilde{s}-1} = \begin{vmatrix} \mathbf{F} & \Phi(s-1) \\ \Psi'^T & 0 \end{vmatrix},$$

$$F_{\tilde{s}-1} = \begin{vmatrix} \mathbf{F} & -\Psi(\tilde{s}-1) \\ \Psi'^T & 1 \end{vmatrix}, \qquad H_{s-1} = \begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) \\ \Phi'^T & 0 \end{vmatrix}, \tag{3.7}$$

$$F_{s+1}^{t+\delta} = \begin{vmatrix} \mathbf{F} & \Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'(-s-1)^T & 1 & 0 \\ \Psi'^T & 0 & 1 \end{vmatrix},$$

$$H^{t+\delta} = \begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'(-s-1)^T & 0 & a_1 & 0 \\ \Phi'^T & 0 & 1 & 0 \\ \Psi'^T & -1 & 0 & 1 \end{vmatrix}.$$

$$G^{t+\delta} = \begin{vmatrix} \mathbf{F} & \Phi(s-1) & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Psi'(-\tilde{s}-1)^T & 0 & 0 & b_1 \\ \Phi'^T & -1 & 1 & 0 \\ \Psi'^T & 0 & 0 & 1 \end{vmatrix},$$

$$H_{\tilde{s}+1}^{t+\delta} = -\begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & \Psi^{l+b_1} \\ \Phi'(-s-1)^T & a_1 & 0 \\ \Phi'^T & 1 & 0 \end{vmatrix}.$$

Then the functions F, G and H given by equations (3.1), (3.2) satisfy the discrete DS system as by substituting equations (3.7) into equations (1.16) we get the following three Jacobi identities for the determinants:

$$|\mathbf{F}| \begin{vmatrix} \mathbf{F} & \Phi & \Psi \\ \Phi'(-s-1)^T & 1 & 0 \\ \Psi'(-\tilde{s}-1)^T & 0 & 1 \end{vmatrix} - \begin{vmatrix} \mathbf{F} & \Phi \\ \Phi'(-s-1)^T & 1 \end{vmatrix} \begin{vmatrix} \mathbf{F} & \Psi \\ \Psi'(-\tilde{s}-1)^T & 1 \end{vmatrix}$$
$$- \begin{vmatrix} \mathbf{F} & \Phi \\ \Psi'(-\tilde{s}-1)^T & 0 \end{vmatrix} \begin{vmatrix} \mathbf{F} & \Psi \\ \Phi'(-s-1)^T & 0 \end{vmatrix} = 0,$$

$$|\mathbf{F}|\begin{vmatrix} \mathbf{F} & \Phi(s-1) & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Psi'(-\tilde{s}-1)^T & 0 & 0 & b_1 \\ \Phi'^T & -1 & 1 & 0 \\ \Psi'^T & 0 & 0 & 1 \end{vmatrix}$$

$$-\begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'^T & 1 & 0 \\ \Psi'^T & 0 & 1 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(s-1) \\ \Psi'(-\tilde{s}-1)^T & 0 \end{vmatrix}$$

$$+a_1\begin{vmatrix} \mathbf{F} & \Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Psi'(-\tilde{s}-1)^T & 0 & b_1 \\ \Psi'^T & 0 & 1 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(s-1) \\ \Phi'^T & -1 \end{vmatrix}$$

$$-b_1\begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & \Psi^{l+b_1} \\ \Psi'(-\tilde{s}-1)^T & 0 & 1 \\ \Phi'^T & 1 & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(s-1) \\ \Psi'^T & 0 \end{vmatrix} = 0, \tag{3.8}$$

$$|\mathbf{F}|\begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'(-s-1)^T & 0 & a_1 & 0 \\ \Phi'^T & 0 & 1 & 0 \\ \Psi'^T & -1 & 0 & 1 \end{vmatrix}$$

$$-\begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'^T & 1 & 0 \\ \Psi'^T & 0 & 1 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) \\ \Phi'(-s-1)^T & 0 \end{vmatrix}$$

$$+a_1\begin{vmatrix} \mathbf{F} & \Phi^{k+a_1} & b_1\Psi^{l+b_1} \\ \Phi'(-s-1)^T & 1 & 0 \\ \Psi'^T & 0 & 1 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) \\ \Phi'^T & 0 \end{vmatrix}$$

$$-b_1\begin{vmatrix} \mathbf{F} & a_1\Phi^{k+a_1} & \Psi^{l+b_1} \\ \Phi'(-s-1)^T & a_1 & 0 \\ \Phi'^T & 1 & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(\tilde{s}-1) \\ \Psi'^T & -1 \end{vmatrix} = 0.$$

□

4. Bilinear Bäcklund transformation and Lax pair

Here, we construct a bilinear Bäcklund transformation for the bilinear equations (1.16), and then we derive from it a Lax pair for the difference DS system (1.18).

To do so, we redefine the functions F, G and H in terms of one function f' depending on an additional discrete variable m:

$$F(s,\tilde{s};t) = f'(m,s,\tilde{s};t), \qquad G(s,\tilde{s};t) = f'(m+1,s,\tilde{s};t),$$
$$H(s,\tilde{s};t) = f'(m-1,s,\tilde{s};t).$$

Then equations (1.16) can be written as

$$[2\,\mathrm{e}^{1/2D_m}\sinh(\delta/2D_t) + b_1\,\mathrm{e}^{D_{\tilde{s}}-1/2D_m+\delta/2D_t} - a_1\,\mathrm{e}^{D_s+1/2D_m+\delta/2D_t}]f'\cdot f' = 0, \tag{4.1}$$
$$[\mathrm{e}^{1/2(D_{\tilde{s}}-D_s)} - \mathrm{e}^{1/2(D_{\tilde{s}}+D_s)} + \mathrm{e}^{1/2(D_s-D_{\tilde{s}})+D_m}]f'\cdot f' = 0.$$

We can now state the following proposition:

Proposition 2. *The bilinear system (4.1) has the Bäcklund transformation*

$$\begin{aligned}
&[\beta_1\,\mathrm{e}^{1/2D_{\tilde{s}}} - \mathrm{e}^{-1/2D_s} - \mu_1\,\mathrm{e}^{D_m-1/2D_{\tilde{s}}}]f'\cdot f'' = 0,\\
&[\beta_2\,\mathrm{e}^{1/2(D_m+D_s)} - \mathrm{e}^{-1/2(D_m+D_s)} + \mu_1\,\mathrm{e}^{1/2(D_m-D_s)}]f'\cdot f'' = 0,\\
&\left[\mathrm{e}^{-\delta/2D_t} + b_1\frac{\mu_1}{\beta_1}\mathrm{e}^{D_m-D_{\tilde{s}}-\delta/2D_t} + a_1\frac{\mu_1}{\beta_2}\mathrm{e}^{-D_s-\delta/2D_t} + \nu\,\mathrm{e}^{\delta/2D_t}\right]f'\cdot f'' = 0.
\end{aligned}\tag{4.2}$$

Proof. Let f' be a solution of equations (4.1). Using equations (4.2), we can, by a straightforward calculation, show that equations (4.1) are satisfied for $f''(m, s, \tilde{s}; t)$:

$$\begin{aligned}
&-[\mathrm{e}^{1/2D_m-\delta/2D_t} f'\cdot f'][2\,\mathrm{e}^{1/2D_m}\sinh(\delta/2D_t) + b_1\,\mathrm{e}^{D_{\tilde{s}}-1/2D_m+\delta/2D_t} - a_1\,\mathrm{e}^{D_s+1/2D_m+\delta/2D_t}]f''\cdot f''\\
&=\{[2\,\mathrm{e}^{1/2D_m}\sinh(\delta/2D_t)+b_1\,\mathrm{e}^{D_{\tilde{s}}-1/2D_m+\delta/2D_t} - a_1\,\mathrm{e}^{D_s+1/2D_m+\delta/2D_t}]f'\cdot f'\}[\mathrm{e}^{1/2D_m-\delta/2D_t} f''\cdot f'']\\
&-\{[2\,\mathrm{e}^{1/2D_m}\sinh(\delta/2D_t)+b_1\,\mathrm{e}^{D_{\tilde{s}}-1/2D_m+\delta/2D_t}-a_1\,\mathrm{e}^{D_s+1/2D_m+\delta/2D_t}]f''\cdot f''\}[\mathrm{e}^{1/2D_m-\delta/2D_t} f'\cdot f']\\
&= 2\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{-\delta/2D_t} f'\cdot f'')\\
&+2b_1\sinh(1/2(D_{\tilde{s}} - D_m) + \delta/2D_t)(\mathrm{e}^{1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{-1/2D_{\tilde{s}}} f'\cdot f'')\\
&-2a_1\sinh(1/2D_s + \delta/2D_t)(\mathrm{e}^{1/2(D_s+D_m)} f'\cdot f'')\cdot(\mathrm{e}^{-1/2(D_s+D_m)} f'\cdot f'')\\
&= 2\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{-\delta/2D_t} f'\cdot f'')\\
&+2b_1\frac{\mu_1}{\beta_1}\sinh(1/2(D_{\tilde{s}} - D_m)\delta/2D_t)(\mathrm{e}^{D_m-1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{-1/2D_{\tilde{s}}} f'\cdot f'')\\
&+2a_1\frac{\mu_1}{\beta_2}\sinh(1/2D_s + \delta/2D_t)(\mathrm{e}^{1/2(D_m-D_s)} f'\cdot f'')\cdot(\mathrm{e}^{-1/2(D_s+D_m)} f'\cdot f'')\\
&= 2\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{-\delta/2D_t} f'\cdot f'')\\
&+2b_1\frac{\mu_1}{\beta_1}\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{D_m-D_{\tilde{s}}-\delta/2D_t} f'\cdot f'')\\
&+2a_1\frac{\mu_1}{\beta_2}\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{-D_s-\delta/2D_t} f'\cdot f'')\\
&= 2\sinh(1/2D_m)(\mathrm{e}^{\delta/2D_t} f'\cdot f'')\cdot(\mathrm{e}^{-\delta/2D_t} f'\cdot f'' + b_1\frac{\mu_1}{\beta_1}\mathrm{e}^{D_m-D_{\tilde{s}}-\delta/2D_t} f'\cdot f''\\
&+a_1\frac{\mu_1}{\beta_2}\mathrm{e}^{-D_s-\delta/2D_t} f'\cdot f'' + \nu\,\mathrm{e}^{\delta/2D_t} f'\cdot f'') = 0,
\end{aligned}$$

$$\begin{aligned}
&-[\mathrm{e}^{1/2(D_{\tilde{s}}+D_s)} f'\cdot f'][\mathrm{e}^{1/2(D_{\tilde{s}}-D_s)} + \mathrm{e}^{1/2(D_s-D_{\tilde{s}})+D_m} - \mathrm{e}^{1/2(D_{\tilde{s}}+D_s)}]f''\cdot f''\\
&= \{[\mathrm{e}^{1/2(D_{\tilde{s}}-D_s)} + \mathrm{e}^{1/2(D_s-D_{\tilde{s}})+D_m} - \mathrm{e}^{1/2(D_{\tilde{s}}+D_s)}]f'\cdot f'\}[\mathrm{e}^{1/2(D_{\tilde{s}}+D_s)} f''\cdot f'']\\
&-\{[\mathrm{e}^{1/2(D_{\tilde{s}}-D_s)} + \mathrm{e}^{1/2(D_s-D_{\tilde{s}})+D_m} - \mathrm{e}^{1/2(D_{\tilde{s}}+D_s)}]f''\cdot f''\}[\mathrm{e}^{1/2(D_{\tilde{s}}+D_s)} f'\cdot f']\\
&= 2\sinh(-1/2D_s)(\mathrm{e}^{1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{-1/2D_{\tilde{s}}} f'\cdot f'')\\
&+2\sinh(1/2(D_m - D_{\tilde{s}}))(\mathrm{e}^{1/2(D_m+D_s)} f'\cdot f'')\cdot(\mathrm{e}^{-1/2(D_m+D_s)} f'\cdot f'')\\
&= -2\mu_1\sinh(1/2D_s)(\mathrm{e}^{1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{D_m-1/2D_{\tilde{s}}} f'\cdot f'')\\
&-2\mu_1\sinh(1/2(D_m - D_{\tilde{s}}))(\mathrm{e}^{1/2(D_m+D_s)} f'\cdot f'')\cdot(\mathrm{e}^{1/2(D_m-D_s)} f'\cdot f'')\\
&= -2\mu_1\sinh(1/2D_s)(\mathrm{e}^{1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{D_m-1/2D_{\tilde{s}}} f'\cdot f'')\\
&+2\mu_1\sinh(1/2D_s)(\mathrm{e}^{1/2D_{\tilde{s}}} f'\cdot f'')\cdot(\mathrm{e}^{D_m-1/2D_{\tilde{s}}} f'\cdot f'') = 0.
\end{aligned}$$

In this way, proposition 2 is proven and equations (4.2) constitute a BT for (4.1). □

From the bilinear Bäcklund transformation (4.2), we can derive a Lax pair for the discrete DS system (1.18) by setting

$$v = \frac{f'^{t+\delta}_{s+1} f'_{s-1}}{f'^{t+\delta} f'}, \qquad \tilde{v} = \frac{f'^{t+\delta}_{\tilde{s}+1} f'_{\tilde{s}-1}}{f'^{t+\delta} f'}, \qquad u = \frac{f'_{m+1}}{f'}, \qquad \tilde{u} = \frac{f'_{m-1}}{f'}, \qquad \phi = \frac{f''}{f'}. \tag{4.3}$$

Under the dependent variable transformation (4.3), the bilinear BT (4.2) becomes the following linear difference equations for ϕ:

$$\begin{aligned} &\beta_1 \phi_{\tilde{s}-1} - \phi - \mu_1 u_{\tilde{s}-1} \tilde{u} \phi_{m-1} = 0, \\ &\beta_2 \tilde{u} \phi_{m-1} - \tilde{u} \phi_{s+1} + \mu_1 \tilde{u}_{s+1} \phi_{m-1,s+1} = 0, \\ &u\phi + \frac{b_1}{\beta_1} u^{t-\delta}_{\tilde{s}-1} \tilde{v}^{t-\delta} (\beta_1 \phi - \phi_{\tilde{s}+1}) + a_1 \frac{\mu_1}{\beta_2} v^{t-\delta} u \phi_{s+1} + \nu u \phi^{t-\delta} = 0, \end{aligned} \tag{4.4}$$

where β_1, β_2, μ_1 are the arbitrary constants. Eliminating $\phi_{m-1,s+1}, \phi_{m-1}$ from equations (4.4), we obtain the following Lax pair for the difference analogue of the DS system (1.18):

$$\beta_2 \left(\frac{\beta_1 \phi_{\tilde{s}-1} - \phi}{\mu_1 u_{\tilde{s}-1}} \right) + \frac{\beta_1 \phi_{\tilde{s}-1,s+1} - \phi_{s+1}}{u_{\tilde{s}-1,s+1}} - \tilde{u} \phi_{s+1} = 0, \tag{4.5}$$

$$u\phi + \frac{b_1}{\beta_1} u^{t-\delta}_{\tilde{s}-1} \tilde{v}^{t-\delta} (\beta_1 \phi - \phi_{\tilde{s}+1}) + a_1 \frac{\mu_1}{\beta_2} v^{t-\delta} u \phi_{s+1} + \nu u \phi^{t-\delta} = 0. \tag{4.6}$$

By imposing the compatibility of equations (4.5), (4.6) we reobtain the discrete DS system (1.18). In fact, from equation (4.5), we derive the expressions of $\phi_{s+1,\tilde{s}-1}, \phi_{s+1,\tilde{s}+1}$ in terms of $\phi, \phi_{\tilde{s}-1}, \phi_{\tilde{s}+1}, \phi_{s+1}$,

$$\beta_1 \phi_{s+1,\tilde{s}-1} = \phi_{s+1} - v_{s+1,\tilde{s}-1} \left[\frac{\beta_2}{\mu_1 u_{\tilde{s}-1}} (\beta_1 \phi_{\tilde{s}-1} - \phi) - \phi_{s+1} \tilde{u} \right], \tag{4.7}$$

$$\mu_1 \phi_{s+1,\tilde{s}+1} = \frac{\beta_2 \left(\beta_1 \phi - \phi_{\tilde{s}+1} + \frac{\beta_1 \mu_1}{\beta_2} \frac{u \phi_{s+1}}{u_{s+1}} \right)}{u(\tilde{u}_{\tilde{s}+1} + 1/u_{s+1})}. \tag{4.8}$$

Shifting equation (4.5) by $-\delta$ in the variable t and substituting it into equations (4.6)–(4.8), we obtain an equation written only in terms of $\phi_{\tilde{s}+1}, \phi_{s+2}, \phi_{\tilde{s}-1}, \phi_{s+1}, \phi$. Equating to zero the coefficients of $\phi_{\tilde{s}+1}, \phi_{s+2}, \phi_{\tilde{s}-1}, \phi_{s+1}, \phi$, we derive the difference equations (1.18).

It is interesting to note here that the spectral problem (4.5) is the same as the one obtained in [18] for the differential-difference DS system (1.1). So both equations (1.18) and (1.10) are associated with the same spectral problem and are thus members of the same hierarchy.

5. Conclusion and discussions

Integrable time discretizations of a differential-difference DS system are considered and turn out to be two different members of a discrete hierarchy of multidimensional equations. One of these is just the Bäcklund transformation for the differential-difference DS system, for which the solutions have been presented in [18]. Here, we obtain the discrete Gram-type determinant solution for the new difference DS system and clarify its bilinear structure in terms of Jacobi identities for determinants. At the end, we present a bilinear Bäcklund transformation for the difference system which enables us to derive its Lax pair. It would be of interest to elucidate some other properties of the difference DS system, such as the relation with numerical algorithms and with Painlevé equations.

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